

## Exercise 44

Find the derivative of the function.

$$y = 2^{3^{4^x}}$$

### Solution

Rewrite the function first.

$$\begin{aligned} y &= e^{\ln 2^{3^{4^x}}} = e^{3^{4^x} \ln 2} = e^{e^{\ln(3^{4^x} \ln 2)}} = e^{e^{\ln 3^{4^x} + \ln(\ln 2)}} = e^{e^{4^x \ln 3 + \ln(\ln 2)}} \\ &= e^{e^{4^x \ln 3} e^{\ln(\ln 2)}} \\ &= e^{e^{4^x \ln 3} (\ln 2)} \\ &= e^{e^{\ln(4^x \ln 3)} (\ln 2)} \\ &= e^{e^{\ln 4^x + \ln(\ln 3)} (\ln 2)} \\ &= e^{e^{e^x \ln 4 + \ln(\ln 3)} (\ln 2)} \\ &= e^{e^{e^x \ln 4} e^{\ln(\ln 3)} (\ln 2)} \\ &= e^{e^{e^x \ln 4} (\ln 3) (\ln 2)} \end{aligned}$$

Then take the derivative using the chain rule.

$$\begin{aligned} y' &= \frac{dy}{dx} = \frac{d}{dx} \left[ e^{e^{e^x \ln 4} (\ln 3) (\ln 2)} \right] \\ &= e^{e^{e^x \ln 4} (\ln 3) (\ln 2)} \cdot \frac{d}{dx} \left[ e^{e^x \ln 4} (\ln 3) (\ln 2) \right] \\ &= e^{e^{e^x \ln 4} (\ln 3) (\ln 2)} \cdot (\ln 2) \frac{d}{dx} \left[ e^{e^x \ln 4} (\ln 3) \right] \\ &= e^{e^{e^x \ln 4} (\ln 3) (\ln 2)} \cdot (\ln 2) \left[ e^{e^x \ln 4} (\ln 3) \right] \cdot \frac{d}{dx} \left[ e^{e^x \ln 4} (\ln 3) \right] \\ &= e^{e^{e^x \ln 4} (\ln 3) (\ln 2)} \cdot (\ln 2) \left[ e^{e^x \ln 4} (\ln 3) \right] \cdot (\ln 3) \frac{d}{dx} \left( e^{e^x \ln 4} \right) \\ &= e^{e^{e^x \ln 4} (\ln 3) (\ln 2)} \cdot (\ln 2) \left[ e^{e^x \ln 4} (\ln 3) \right] \cdot (\ln 3) \left( e^{e^x \ln 4} \right) \cdot \frac{d}{dx} (e^{e^x \ln 4}) \\ &= e^{e^{e^x \ln 4} (\ln 3) (\ln 2)} \cdot (\ln 2) \left[ e^{e^x \ln 4} (\ln 3) \right] \cdot (\ln 3) \left( e^{e^x \ln 4} \right) \cdot (\ln 4) \\ &= (\ln 2)(\ln 3)(\ln 4) e^{e^{e^x \ln 4} (\ln 3) (\ln 2)} e^{e^x \ln 4} (\ln 3) e^{e^x \ln 4} \\ &= (\ln 2)(\ln 3)(\ln 4) 2^{3^{4^x}} 3^{4^x} 4^x \end{aligned}$$